

Component Ratios of Independent and Herding Betters in a Racetrack Betting Market

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We study the time series data of the racetrack betting market in the Japan Racing Association (JRA). As the number of votes t increases, the win bet fraction $x(t)$ converges to the final win bet fraction x_f . We observe the power law $(x(t) - x_f)^2 \propto t^{-\beta}$ with $\beta \simeq 0.488$. We measure the degree of the completeness of the ordering of the horses using an index AR, the horses are ranked according to the size of the win bet fraction. AR(t) also obeys the power law and behaves as $AR_f - AR(t) \propto t^{-\gamma}$ with $\gamma \simeq 0.589$, where AR_f is the final value of AR. We introduce a simple voting model with two types of voters—-independent and herding. Independent voters provide information about the winning probability of the horses and herding voters decide their votes based on the popularities of the horses. This model can explain two power laws of the betting process. The component ratio of the independent voter to the herding voter is 1:3.

KEYWORDS: power law, betting market, stochastic process, accuracy ratio, efficiency, herding

1. Introduction

Racetrack betting is a simple exercise of gaining a profit or losing one's wager. However, one needs to make a decision in the face of uncertainty, and a closer inspection of this decision-making process reveals great complexity and scope. The field has attracted many academics from various disciplines and has become a subject of wider importance.¹⁾ Compared to the stock or currency exchange markets, racetrack betting is a short-lived and repeated market. It is possible to obtain a clearer understanding of aggregated betting behaviour and study the market efficiency. One of the main findings of the previous studies is the 'favorite-longshot bias' in the racetrack betting market.^{2,3)} The final odds are, on average, accurate measures of winning, short-odds horses are systematically undervalued, and long-odds horses are systematically overvalued.

From an econophysical perspective, racetrack betting is an interesting subject. Park and Dommany have analysed the distribution of the final odds (dividends) of the races organised by the Korean Racing Association.⁴⁾ They observed that the distribution of the final odds

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exhibited the power law behaviour. They explained this behaviour on the basis of the assumptions of a rational better who maximizes the expected payoff and a multiplicative rule in the estimate of the winning probability. Ichinomiya also observed the power law of final odds in the races organised by the Japan Racing Association (JRA)⁵⁾ and proposed another betting model where the strength of the horse obeys uniform distribution and the complex competition process in the horse race is described by normal distribution. He also assumed that betters exhibit irrational behaviour and bet their money on the horse that appears to be strongest in the race. The authors analysed the uncertainty in the prediction of the racetrack betting market. It is a short-lived and repeated market, and hence the accuracy of the predictions can be estimated. We found a scale-invariant relation between the rank of a racehorse and the result of its victory or defeat in JRA.⁶⁾ Horses are ranked according to the win bet fractions. In the long-odds region, between the cumulative distribution function of the winning horses x_1 and that of the losing horses x_0 , a scale-invariant relation $x_1 \propto x_0^\alpha$ with $\alpha = 1.81$ holds. In a betting model where betters display herd-like behaviour (herding better) with only a small amount of information about the strength of the horses and vote on the horses according to the probabilities that are proportional to the number of the votes, it is possible to show that the scale invariance emerges in a self-organized fashion. The authors also studied another betting model with two types of voters—-independent and herding.⁷⁾ Independent voters provide information about the winning probability of the horses to the market. It was found that a phase transition occurs in the process of information aggregation and that herding voters are responsible for the slow convergence of the win bet fraction.

Summarizing these studies, we are faced with two questions about the racetrack betting market. The first question is whether the betters are rational or not. The final odds contain an accurate estimate of the winning probability, which means that the betters look rational. However, it is improbable that all betters are clever and are able to precisely estimate the winning probability. In the exchange or stock market, the role of fundamental and chartist-type participants has been discussed.⁹⁾ The component ratios of the two types of participants change drastically, and thus, the market exhibits a complex behavior. It was also discussed that herding voters increase the accuracy of prediction in a forecasting game, and such herding behaviour may be very efficient in aggregating dispersed private information.¹⁰⁾ In the racetrack betting market, whether such types of participants exist is an interesting question.

The second question is the distribution of final odds, which reflects the winning probability of a horse. Previous studies have proposed the two possibility of potential mechanisms. The drawback of Park and Dommany's model is that the true winning probability distribution does not come from the betting or estimating process. It instead comes from the system in which many horses run at the same time and try to get to the first position after complex competition process. If the multiplicative estimation rule produces the empirical distribution

of the winning probability, the model can represent the complex competition process. Betterers understand how to estimate the winning probability after studying many horse races; the final odds reflect the true winning probability. Although Ichinomiya proposed an interesting mechanism to observe power law in the final odds, his assumption of irrational betterers cannot be accepted. If betterers are not rational, it is difficult to understand the efficiency of the market where the win bet fraction coincides with the true winning probability. In this study, we focus on the rationality of the betterers. We study the time series data of the win bet in the JRA and the nature of the betterers, i.e., whether they are rational or not, and what type of betterer exists in the racetrack betting market. With respect to the distribution of the final odds, we only assume that the winning probability has a broad distribution and can be estimated on the basis of the final odds.

The organization of this paper is as follows. In §2, we provide a detailed study of the time series data of the win bet. We have studied the time series data of the win bet odds in 2008 of JRA. Horses are ranked according to the win bet fraction and the receiver operation characteristic (ROC) curve is discussed. We measure the fluctuation of the win bet fraction x and the degree of the completeness of the ordering of the horses by an index AR with the progress of the voting. In §3, we show the result of the data analysis. As the number of vote t increases, $x(t)$ converges to the final values x_f very slowly. The power law relation $(x(t) - x_f)^2 \propto t^{-\beta}$ with $\beta \simeq 0.488$ holds. $\text{AR}(t)$ also obeys a power law $\text{AR}_f - \text{AR}(t) \propto t^{-\gamma}$ with $\gamma \simeq 0.589$, where AR_f is the final value of AR. In §4, we introduce a voting model, where there are two types of voters—-independent and herding. Using the exponent in the power law convergence of $x(t)$, we estimate the component ratio of the independent voter to the herding voter is 1:3. The power law convergence of $\text{AR}(t)$ is also observed in this voting model. Section 5 is dedicated to the summary.

2. Racetrack Betting Process

We study the win bet data of JRA races in 2008. A win bet is the wager that the betterer lays on the winner of the race. Out of the 3542 in that year, we choose 2471 races whose final public win pool (total number of votes) V^r is in the range of $10^5 \leq V^r \leq 3 \times 10^5$, $r \in \{1, 2, \dots, R = 2471\}$. The average value of V^r is about $1.89 \cdot 10^5$. N^r horses run in race r ; N^r is in the range of $7 \leq N^r \leq 18$. We ignore 102 canceled horses; the total number of horses $N \equiv \sum_{r=1}^R N^r$ is 35719. The number of the winning horse is 2472 (one tie occurs) and is denoted as $N_1 = 2472$. The number of the remaining horses (losing horses) is denoted as $N_0 = N - N_1$. K^r denotes the number of times a public announcement was made regarding the temporal odds and number of votes in race r . K^r is in the range of $13 \leq K^r \leq 217$, and the total number of announcements is $K \equiv \sum_{r=1}^R K^r \simeq 2.0 \times 10^5$. On an average, announcements were made eighty times up till the start of the races. The time from the announcement to the race entry time (start of the race) in minutes is denoted as T_k^r . We denote the temporal

odds of the i th horse in race r at the k th announcement as $O_{i,k}^r$; the public win pool as V_k^r . $V_{K^r}^r = V^r$ holds. I_i^r denotes the results of the races. $I_i^r = 1(0)$ implies that horse i wins (loses) in race r . A typical sample from the data is shown in Table I.

Table I. Time series of odds and pool for a race that starts at 13 : 00. $N^r = 10$, $K^r = 52$. We show the data only for the first three horses $1 \leq i \leq 3$. The first horse wins the race ($I_1^r = 1$, $I_2^r = I_3^r = \dots = 0$).

k	T_k^r [min]	V_k^r	$O_{1,k}^r$	$O_{2,k}^r$	$O_{3,k}^r$	\dots
1	358	1	0.0	0.0	0.0	\dots
2	351	169	1.6	33.3	7.9	\dots
3	343	314	1.8	11.3	8.0	\dots
4	336	812	2.9	17.8	14.6	\dots
5	329	1400	3.3	8.6	10.6	\dots
6	322	1587	2.7	9.2	11.3	\dots
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
51	10	80064	2.4	6.4	13.4	\dots
52	4	148289	2.4	4.9	16.1	\dots
53	-2	211653	2.4	5.3	17.0	\dots

From $O_{i,k}^r$, we estimate the win bet fraction $x_{i,k}^r$ by the following relation according to the rule by JRA.

$$x_{i,k}^r = \frac{0.788}{O_{i,k}^r - 0.1}. \quad (1)$$

If the sum of the above values does not equal 1 in each announcement, we renormalize it as $\hat{x}_{i,k}^r = \frac{x_{i,k}^r}{\sum_{i=1}^{N^r} x_{i,k}^r}$. Hereafter, we use $x_{i,k}^r$ in place of $\hat{x}_{i,k}^r$.

We use the public win pool averaged over all the races as the time variable t for the entire betting process. For each $0 \leq v \leq 3 \times 10^5$, we select the nearest $V_{i,k}^r$ and use the average value of $V_{i,k}^r$ as the time variable t . More explicitly, we define t as

$$t(v) \equiv \frac{1}{N} \sum_{r=1}^R V_{k^r(v)}^r \cdot N^r, \quad (2)$$

$$k^r(v) \equiv \{k \mid \text{Min}_k |V_k^r - v|\}. \quad (3)$$

The range of t is $70 \leq t \leq 1.89 \times 10^5$; the largest time is denoted as $t_f \equiv 1.89 \times 10^5$. At t_f , the number of votes v becomes V^r ; t_f represents the end of the voting process. If t exceeds 10^5 , it cannot be accurately regarded as a time variable. As the public win pool V^r is in the range $10^5 - 3 \times 10^5$, if t exceeds the V^r of some races, then the voting ends for those races. The results of the data analysis for $t \geq 10^5$ does not provide true information about the time

evolution of the voting process. The data for $t \geq 10^5$ is provided only for the purpose of reference. We also define $x_i^r(t)$ as

$$x_i^r(t) \equiv x_{i,k^r(v)}^r. \quad (4)$$

The average value of T_k^r is denoted as $T(t)$ and defined as

$$T(t) \equiv \frac{1}{N} \sum_{r=1}^R T_{k^r(v)}^r \cdot N^r. \quad (5)$$

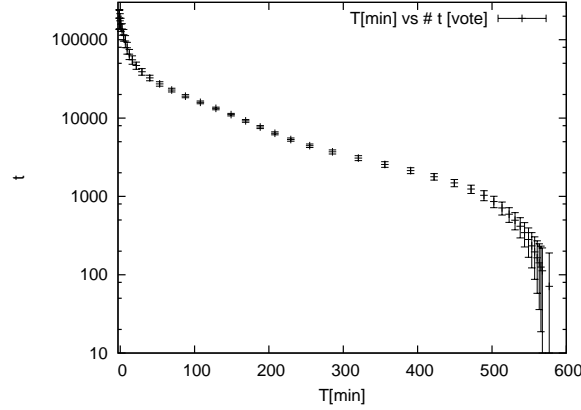


Fig. 1. Relationship between $T[\text{min}]$ and average number of votes t . In the first announcement ($k = 1$), which occurs approximately 10 h before the start of the race, the average number of votes t is 71.0. Approximately 30 min before the start of the races, about 4×10^4 votes have been cast.

Figure 1 shows the relationship between T and t . A rapid growth is observed in the average number of votes t as we approach the start of the race ($T \rightarrow 0$). Almost half of the votes are thrown in the last 9 min.

In order to provide a pictorial representation of the betting process pictorially, we arrange the N horses in the order of the size of $x_i^r(t)$. We denote the arranged win bet fraction as $x_\alpha(t)$, $\alpha \in \{1, 2, \dots, N\}$.

$$x_1(t) \geq x_2(t) \geq x_3(t) \geq \dots \geq x_N(t) \quad (6)$$

$I_\alpha(t)$ tells us whether horse α wins ($I_\alpha = 1$) or loses ($I_\alpha = 0$). In general, the probability that the horse with a large $x_\alpha(t)$ wins is big and vice versa. We arrange the horses in the increasing order of α from left to right. The left-hand side of the sequence represents stronger horses, and the right-hand side, the weaker horses. If the win bet fraction does not contain any information about the strength of the horses, $I_\alpha(t)$ randomly assumes the value of one and zero. Conversely, if the information is completely correct, the first N_1 horses' $I_\alpha(t)$ are 1 and the remaining N_0 horses' $I_\alpha(t)$ are 0. In general, as t increases, the accuracy of $x_\alpha(t)$ increases and the strong horses with large winning probabilities move to the left and vice versa.

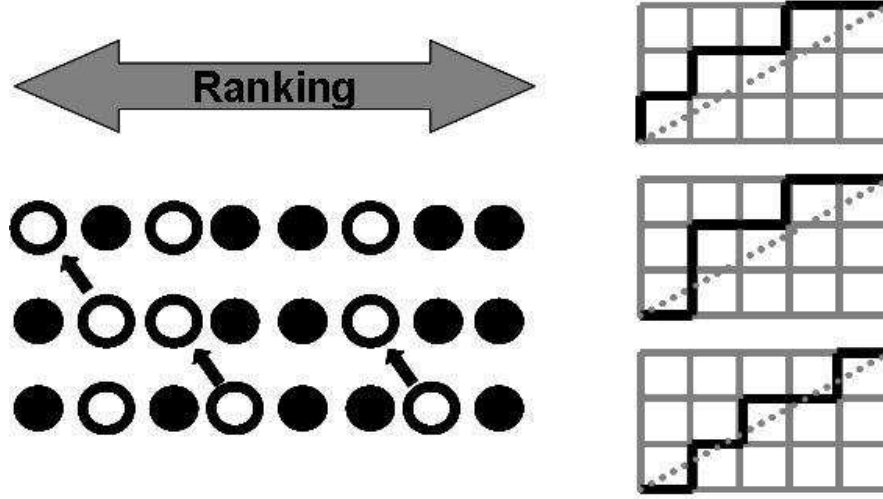


Fig. 2. Pictorial representation of the movement of the ranking of the horses by betting process. Three horses are winning and 5 horses are losing. As t increases, we move from the bottom row to the top one. We depict the winning (losing) horses by white (black) circles. On the right, we show the corresponding ROC curve.

Figure 2(left) shows the movement of the ranking of the horses due to voting. There are eight horses; three of them are winning ones (white circle) and the remaining five are losing ones (black circle). The initial configuration, which corresponds to the first announcement $k = 1$ in each race, is shown in the bottom row. In terms of ranking, the 2nd, 4th, and 7th horses are winning and the remaining horses are losing ($I_2 = I_4 = I_7 = 1, I_1 = I_3 = I_5 = I_6 = I_8 = 0$). As t increases, the rank of the winning horses moves to the left in general and the accuracy of the prediction by the betters is improved. At the final state, the 1st, 3rd, and 6th horses are winning ones ($I_1 = I_3 = I_6 = 1, I_2 = I_4 = I_5 = I_7 = I_8 = 0$).

We employ the receiver operating characteristic (ROC) curve¹¹⁾ and observe the movement of the ranking and the increase in the accuracy more pictorially. It is a path $\{(x_{0,k}, x_{1,k})\}_{k=0, \dots, N}$ in two-dimensional space (x_0, x_1) from $(x_{0,0}, x_{1,0}) = (0, 0)$ to $(x_{0,N}, x_{1,N}) = (1, 1)$ as

$$x_{\mu,k} = \frac{1}{N_{\mu}} \sum_{j=1}^k \delta_{I_j, \mu}. \quad (7)$$

If $I_k = \mu$, the path extends in x_{μ} direction. If the winning and losing horses are sufficiently mixed in the ranking space, the path almost runs diagonally to the end point. If the accuracy of the prediction is good, they are separated and the path resembles an upward convex curve from $(0, 0)$ to $(1, 1)$. Figure 2(right) shows the ROC curves corresponding to the ranking configuration $\{I_{\alpha}\}_{\alpha=1, \dots, N}$ on the left. For the bottom case, the win bet fractions x_{α} do not contain sufficient information about the strength of the horses. The ROC curve is almost

along the diagonal line. As the betting progresses from bottom to top, the phase separation between the two categories of the horses does occur and ROC curves become more and more upwardly convex.

We are able to discuss the discriminative power of the better on the basis of the probability that the ranking of a randomly selected winning horse α_w is higher than that of a randomly selected losing one α_l .¹¹⁾ The normalized index called accuracy ratio AR is defined as

$$\text{AR} \equiv 2 \cdot (\text{Prob}(\alpha_w < \alpha_l) - \frac{1}{2}). \quad (8)$$

If the betters cannot make any discrimination, the horses are mixed randomly. $\text{Prob}(\alpha_w < \alpha_l)$ becomes $\frac{1}{2}$ and AR becomes zero. If the betters can make a strong (or a completely accurate) discrimination, both $\text{Prob}(\alpha_w > \alpha_l)$ and AR become 1. $\text{Prob}(\alpha_w < \alpha_l)$ is also the area below the ROC curve, and AR can be estimated as

$$\text{AR} = 2 \cdot \left(\sum_{k=1}^N x_{1,k} \cdot (x_{0,k} - x_{0,k-1}) - \frac{1}{2} \right). \quad (9)$$

AR changes from 1/15 to 5/15 to 7/15 from the bottom row to the top row in Fig 2.

3. Power Law Convergence of $x_\alpha(t)$ and $\text{AR}(t)$

In this section, we explain the results of the analysis of the time series. We start from the convergence of the win bet fraction $x_\alpha(t)$ to its final value $x_{\alpha,f}$, where $x_{\alpha,f}$ is the final value of the win bet fraction $x_{\alpha,f} \equiv x_\alpha(t_f)$. $x_{\alpha,f}$ is the winning probability of the horse α agreed by all the betters who vote in the race. It is the subjective probability or the risk neutral probability of the victory of the horse. We cannot compute the true winning probability (objective winning probability) of the horse. If several horses with almost the same value of the win bet fraction x_f are grouped together, the winning rate of the horses coincides with the win bet fraction.^{1,8)} In this manner, the market is shown to be efficient and no one can get surplus gain by knowing the value of x_f .

We calculate the average value of the squared fluctuation over the horses as

$$[(x_\alpha(t) - x_{\alpha,f})^2] \equiv \frac{1}{N} \sum_{\alpha=1}^N (x_\alpha(t) - x_{\alpha,f})^2. \quad (10)$$

If the voting has been performed by voters independently, $[(x_\alpha(t) - x_{\alpha,f})^2]$ depends on t as t^{-1} ; this is termed as normal diffusion. If the behaviour deviates from t^{-1} to $t^{-\beta}$ with $\beta < 1$, the power law convergence is termed as super diffusive.¹²⁾

Figure 3 shows the double logarithmic plot of $[(x_\alpha(t) - x_{\alpha,f})^2]$ as a function of t . We observe a slow convergence of $x_\alpha(t)$ to $x_{\alpha,f}$ and a power law behaviour as $t^{-\beta}$ with $\beta = 0.488 \pm 0.007$ for $t \leq t_c \simeq 3 \times 10^4$. After t_c , the convergence occurs rapidly. This sort of super diffusive behaviour has been observed in many types of data such as coarse-grained DNA sequences, written texts, and financial data. Figure 3 also shows the plot of $\text{AR}_f - \text{AR}(t)$ as a function

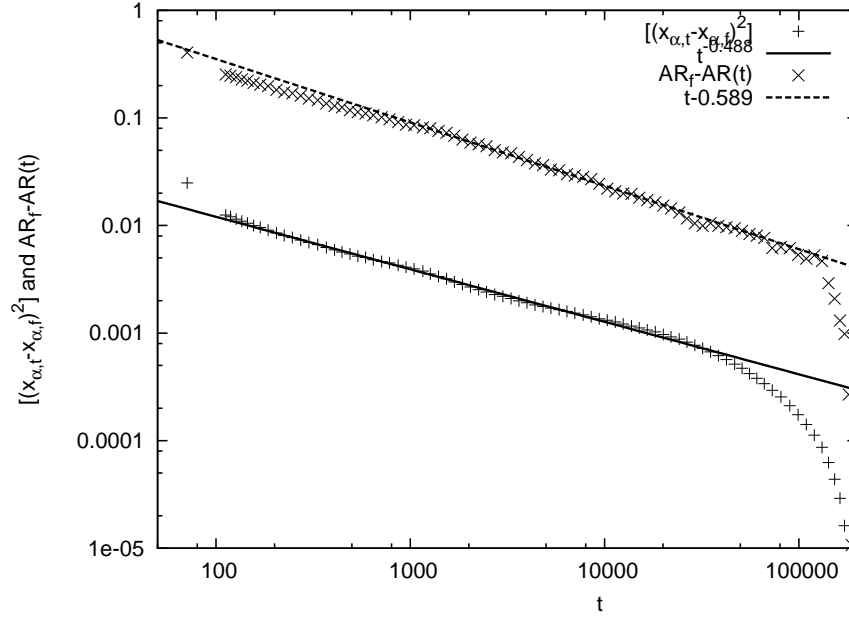


Fig. 3. Double logarithmic plot of $[(x_\alpha(t) - x_{\alpha,f})^2]$ and $AR_f - AR(t)$ as functions of t . The fitted lines with the power law function $a \cdot t^{-\beta}$ are also plotted.

of t , where AR_f is the final value of $AR_f \equiv AR(t_f)$. The fitted curve $AR_f - a \cdot t^{-\gamma}$ with $\gamma = 0.589 \pm 0.005$ and $AR_f = 0.6826$ is also shown. We observe a slow convergence and power law behaviour of $AR(t)$. Contrary to the convergence of $x_\alpha(t)$, the power law relation holds for a wider range of t . We note that after $t = 10^5$, the voting ends in some races with $V^r < t$; in these case the plot does not reflect the true time evolution of the voting process.

4. Voting Model with Independent and Herding Voters

In this section, we introduce a voting model and explain the power law behaviours mentioned in the previous section. There are N^r horses; this number varies among races. We assume it to be a constant value, $N^r = N/R$. Voters vote for the horses individually, and the result of each voting is announced promptly. The time variable $t \in \{0, 1, 2, \dots, T\}$ represents the number of the votes. Among N^r horses, we select a horse with winning probability w and call it the target horse. The probability that any other horse wins is then $1 - w$. Voters somehow know the value w , and after many votes, the win bet fraction coincides with w . We denote the number of votes of the target horse at time t as X_t^w . At $t = 0$, X_t^w takes the initial value $X_0^w = s > 0$. There are N^r horses in a race and the sum of X_t^w is $N^r s + t$. If the target horse gets a vote at t , X_t^w increases by one unit.

$$X_{t+1}^w = X_t^w + 1.$$

We introduce two types of voters—-independent and herding. Independent voters their vote on the basis of their private information and are not affected by the value of X_t^w . These

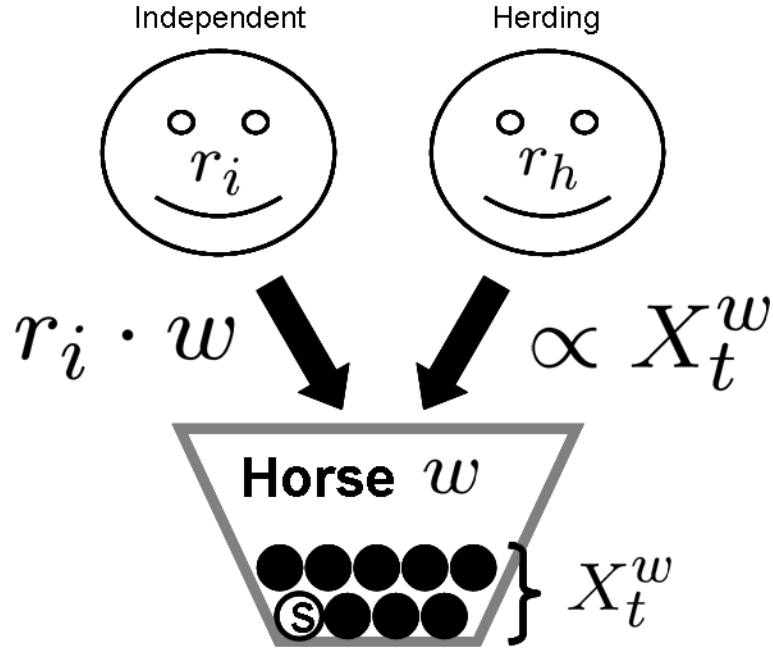


Fig. 4. Representation of a voting model. There are two types of voters—-independent and herding. The component rates are r_i and r_h . The target horse has the winning probability w and the independent voters voting for it is also w . The herding voters decide their vote on the basis of the popularity X_t^w of the horse.

voters provide information about the strength of the horses. We assume that the ratio of the independent voters who vote for the target horse with true winning probability w is w . If there are only independent voters in the market, the win bet fraction $x_t^w \equiv \frac{X_t^w - s}{t}$ converges to the winning probability w according to the power law $\langle (x_t^w - w)^2 \rangle \sim 1/t$. Here, $\langle \rangle$ means the average value over all possible paths of the stochastic voting process. The herding voters decide their vote on the basis of the popularity of the horse and do not rely on private information. A herding voter casts a vote for the target horse at a rate proportional to X_t^w . They do not bring in any information about the horses. They cause slow convergence of the win bet fraction x_t^w to the final value w .⁷⁾

Regarding the rationalities of the voters, we make the following two statements. The herding voters are rational because they have no information about the strength of the horses, and the best way for them is to get information from the results of previous votes. If a horse get many votes, many voters agree that the horse is strong. Hence, the herding voters feel it rational to cast a vote to the popular horse and their behaviour is also rational. Of course, the total votes also include votes cast by herding voters who might provide wrong information. If the ratio of independent voters who vote for a horse with winning probability w coincides with w , the fluctuation induced by the herding voters is cancelled after the votes have been cast many times.

The independent voters seems irrational because they vote for the horse they think might win the race. In addition, the assumption that the ratio of the independent voters coincides with the true winning probability is unrealistic. However, their voting behaviour can be understood to be similar to that of fundamental voters. Fundamental voters are rational in the sense that they vote for the horse with the maximum expected payoff. If the win bet fraction is smaller than the true winning probability, the expected payoff of the horse is larger than the average expected payoff of other horses. We assume that the fundamental voters vote for the target horse with the probability proportional to the difference between the win bet fraction and the true winning probability. The resulting probabilistic rule is then transformed to the rule of the independent and herding voters, as we shall show below. The independent voters can be considered to be rational fundamental voters, if the ratio of the independent voters who vote for the horse with the true winning probability w is equal to w .

We denote the component rates of the independent and herding voters as r_i and r_h , respectively. Obviously, these rates add up to one, i.e., $r_i + r_h = 1$. Figure 4 explains the model pictorially. Mathematically, we can express the above definition of the model using a simple master equation. The probability P_t^w that the target horse gets a vote if the horse get n votes up to t and $X_t^w = n + s$ is

$$P_t^w(X_t^w = n + s) = r_i \cdot w + r_h \cdot \frac{n + s}{Z + t}, \quad (11)$$

$$Z = N^r \cdot s. \quad (12)$$

The probability $P(n, t + 1)$ of finding n votes in $t + 1$ voting times follows the evolution equation

$$P(n, t + 1) = (1 - P_t^w(n))P(n, t) + P_t^w(n - 1)P(n - 1, t). \quad (13)$$

If t is sufficiently large and the win bet fraction is x_t^w , P_t^w is expressed as

$$P_t^w = r_i \cdot w + r_h \cdot x_t^w. \quad (14)$$

In the case of fundamental and herding voters, the probabilistic rule is expressed for a sufficiently large t as

$$P_t^w = r_f \cdot (w + \lambda(w - x_t^w)) + r'_h \cdot x_t^w. \quad (15)$$

Here, we denote the component ratio of fundamental and herding voters as r_f and r'_h , respectively. These voters vote for the target horse with a probability of w if x_t^w coincides with w . If x_t^w is different from w , the probability of voting for the horse changes with $\lambda(w - x_t^w)$ where λ is a proportional coefficient. By comparing eqs (14) and (15), the latter model can be mapped to the former model by the relation,

$$r_i = (1 + \lambda) \cdot r_f \quad \text{and} \quad r_h = r'_h - \lambda r_f. \quad (16)$$

The voting model of independent and herding voters can be considered to be the voting model

of rational voters.

We also assume that w obeys gamma distribution with a shape exponent a and scale parameter c as $p_{a,c}(w)$.

$$p_{a,c}(w) \equiv \frac{1}{c\Gamma(a)} \left(\frac{w}{c}\right)^{a-1} \exp(-w/c). \quad (17)$$

The expected value of w is $[w]_w = c \cdot a$; we take this value to be $1/N^r$. Here, $[A]_w$ is defined as the average of $A(w)$ over $p_{a,c}(w)$ as $[A]_w \equiv \int_0^1 p_{a,c}(w) A(w) dw$. In addition, we fit the distribution of x_f with the gamma distribution $p_{a,c}(w)$ by the least square method and we set the value of $a = 0.47$.⁶⁾ The resultant AR in the model converges to $\text{AR}_f = 0.682$. Hereafter, we concentrate on the power law behaviours of the model, which do not depend on the detailed nature of the distribution of w .

As has been discussed previously,⁷⁾ the win bet fraction x_t^w converges to w after infinite times of voting. The convergence shows the power law behaviour as

$$(x_t^w - w)^2 \sim t^{-1} \quad \text{if } r_i > \frac{1}{2} \quad (18)$$

$$(x_t^w - w)^2 \sim t^{-2r_i} \quad \text{if } r_i < \frac{1}{2} \quad (19)$$

$$(x_t^w - w)^2 \sim \frac{\log(t)}{t} \quad \text{if } r_i = \frac{1}{2}. \quad (20)$$

The power law exponent does not depend on w . After averaging $(x_t^w - w)^2$ over w with $p_{a,c}(w)$, the critical behaviour remains the same. We obtain the exponent β for the convergence as $\beta = 0.488$ and we take r_i to be half of β , i.e., as $r_i = \beta/2 = 0.244$. The only remaining parameter to be set in the voting model is s the initial seeds, i.e., $X_0^w = s$. This parameter describes the strength of the correlation between the votes. In the case where $r_i = 0$, the correlation function is calculated as $1/(Z + 1) = 1/(N^r s + 1)$.⁶⁾ If s is small, the votes are concentrated to particular horses and the variance of the win bet fraction becomes large.

Figure 5 shows the result of the convergence of $x^w(t)$ to x_f . In the figure, we show the double logarithmic plot of $[< (x^w(t) - w)^2 >]_w$ vs t . We set $s = 3$ and 0.3 . We also show the plot of $[(x_\alpha(t) - x_{\alpha,f})^2]$ for comparison. The two former curves are straight lines with a slope of $2r_i = \beta$. Compared to the data plot, the model's curves show power law behaviour up to the end. We also see a large discrepancy between the two plots and the data plot. The variance $[(x_\alpha(t) - x_{\alpha,f})^2]$ is far larger than $[< (x^w(t) - w)^2 >]_w$. By selecting a small s , we can increase the variance of x_t^w and reduce the discrepancy. The votes up to the first announcement $k = 1$ are highly concentrated to one or two horses. AR is very small at $k = 1$, which can be seen by the presence of isolated data points at $t \simeq 70$. The votes provide almost no information up to the first announcement ($k = 1$). The effect of the misleading and concentrated votes remains during the power law convergence period. It is only after t_c that the bias begins to disappear quickly.

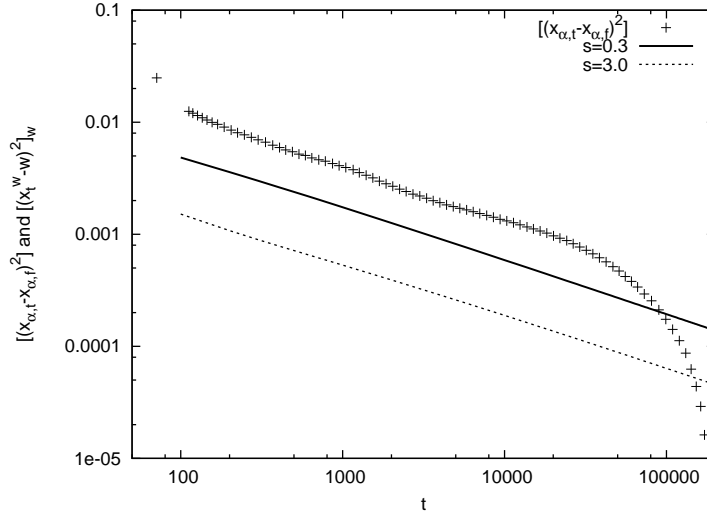


Fig. 5. Double logarithmic plot of the averaged win bet fraction $[(x_t^w - w)^2]_w$ vs t . From bottom, we set $s = 3$ (dotted) and 0.3 (solid). The top line is the plot $[(x_t^w - w)^2]$ vs t .

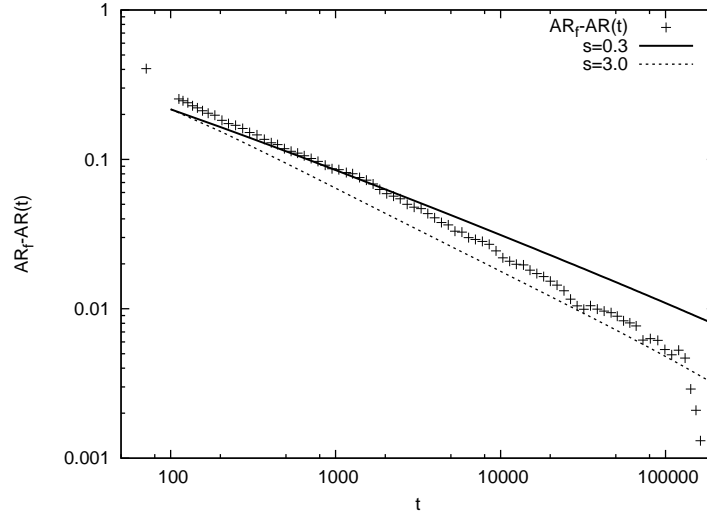


Fig. 6. Double logarithmic plot of the averaged win bet fraction $[(x_t^w - w)^2]_w$ vs t . As in the previous figure, we set $s = 3$ (dotted) and 0.3 (solid). The top line is the plot $AR_f - AR(t)$ vs t .

Figure 6 shows the result of the convergence of $AR(t)$ to AR_f for the same set of the values of s . The double logarithmic plot is straight for $10^2 \leq t \leq 2 \times 10^5$, and the convergence seems to show power law behaviour. Contrary to the convergence of $x(t)$, the slope or the critical exponent of the convergence of AR depends on s . $s = 3$ is a good selection, as observed from the comparison with the curve of data plot.

5. Concluding Remarks

In this paper, we study the time series data of the win bet in JRA. We use the number of votes as the time variable t of the voting process. As functions of t , the win bet fraction $x_\alpha(t)$ and the accuracy of predictions AR obey power laws. We obtain $[(x_\alpha(t) - x_{\alpha,f})^2] \sim t^{-0.488}$ and $\text{AR}_f - \text{AR}(t) \sim t^{-0.589}$. The range where the power law holds is wider in $\text{AR}(t)$ than in $x_\alpha(t)$. After t_c , the convergence of the win bet fraction becomes fast. However, AR obey the power law relation even after t_c , almost up to 10^5 .

We introduce a simple voting model with two types of voters—*independent* and *herding*. The former voters provide information on the strengths of the horses in the racetrack betting market, while latter decide on which horse to vote on the basis of the popularities of the horses. We assume that the component ratio of the independent voters coincides with the true winning probability of the horse that they vote for and that all the independent voters make rational decisions. We also discuss the relationship between the model with independent and herding voters and the model with fundamental and herding voters. We show that the voting model can be used to explain the abovementioned power law behaviours. From the exponent of the convergence of the win bet fraction, it is observed that the component ratio of the independent voter to the herding voter is 1:3.

The change in the convergence after t_c cannot be easily explained by increasing the component ratio of independent voters. If this ratio is increased, the win bet fraction converges more rapidly. However, this increase also causes the rapid convergence of AR, which contradicts with the behaviour of the power law convergence of AR even after t_c . After t_c , the behaviour of betters can possibly change. We believe that the power law of AR holds even after t_c because the component ratio of the voters who provide information to the market does not change. A more detailed analysis of the time series of the betting processes must be performed in the future. In addition, it is also important to study the time dependence of AR. In this study, we have numerically analysed AR and showed that it increases very slowly and seems to obey the power law relation. In contrast to the win bet fraction, the power law dependence has not been investigated mathematically. AR is related to the area under the ROC curve. Statistical properties of the probabilistic growth of the curve induced by voting are an interesting problem and we believe that it should be studied.

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